Circumvention of Haag's theorem in perturbative QFTs

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1 Haag's theorem

Let $\mathcal{A}_0$ be a CCR-algebra of field operators living on the Fock-space $\mathcal{H}_0$ of the free field in Minkowski spacetime. On $\mathcal{H}_0$ is a unitary implementation of the spatial translations, called $U(\vec{x})$. For the free field $\phi_0 \in \mathcal{A}$ we have

(A) equal time CCRs

$$[\phi_0(0,f), \phi_0(0,g)] = i(f,g)\mathbb{1}$$

(B) time-evolution: there exists a (ess.) self-adjoint operator H_0 on $\mathcal{H}_0$, such that

$$\phi_0(t,\vec{x}) = \mathrm{e}^{itH_0}\phi_0(0,\vec{x})\mathrm{e}^{-itH_0}$$

(C) translation covariance : The generator of translations commutes with H:

$$U(\vec{y})\phi_0(t,\vec{x})U(\vec{y})^* = \phi_0(t,\vec{x}+\vec{y})$$

What we at least want from an interacting field, living on $\mathcal{H}_0$:

(A) Identification of field concepts: $\Phi(0,\vec{x}) = \phi_0(0,\vec{x})$ and $\dot{\Phi}(0,\vec{x}) = \dot{\phi}_0(0,\vec{x})$

(B) Interacting time-evolution: There is a ess. self-adjoint H on $\mathcal{H}_0$, such that

$$\Phi(t,\vec{x}) = \mathrm{e}^{itH}\Phi(0,\vec{x})\mathrm{e}^{itH}$$

(C) The generator of translations commutes with H, such that

$$U(\vec{y})\Phi(t,\vec{x})U(\vec{y})^* = \Phi(t,\vec{x}+\vec{y})$$

Theorem 1 (Haag)**.** *If assumptions A-C hold on $\mathcal{H}_0$, then $H = H_0 + const.$.*

The theorem depends heavily on the fact, that the vacuum vector on $\mathcal{H}_0$ is the only translation invariant vector on $\mathcal{H}_0$.

2 Circumvention for exact interacting theories – breaking of translation invariance

2.1 The interacting fields

To circumvent this theorem, one has to get rid of one of the assumptions. The usual way is to break translation invariance, i.e. (C) by replacing H by

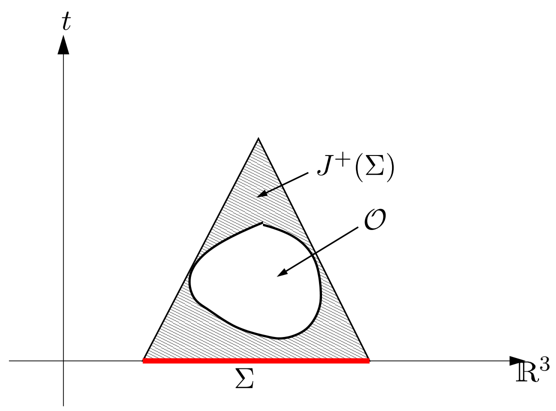
$$H(g) = H_0 + \underbrace{\int \mathrm{d}^3\vec{x}\, g(\vec{x})\, \mathbf{:}V(\Phi(0,\vec{x}))\mathbf{:}}_{V(g)}, \qquad g \in C_0^\infty(\mathbb{R}^3)$$

where $\mathbf{:}V(\Phi(0,\vec{x}))\mathbf{:}$ is supposed to be the (normal ordered) interaction (polynomial in $\Phi(0,\vec{x})$). This has the following advantages:

- takes the distributional character of Wick-polynomials seriously
- wolg. g can be seen as $g \in C_0^\infty(\mathbb{R}^4)$, cutting off the interaction Lagrangian

  $$S_g = \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{int}}(\phi_0(x))g(x)$$

  i.e. it serves as a covariant IR-cutoff for the interaction
- by setting $g = 1$ on a subset Σ of its support, the interacting field Φ is independent of g in the future domain of dependence $J^+(\Sigma)$ of Σ
- the local algebra of interacting fields $\mathcal{A}_V(\mathcal{O})$ on $\mathcal{O}$ are then generated by the interacting fields $\Phi(t,x)$, by setting $g = 1$ on a sufficiently large region, such that: $J^+(\Sigma) \supset \mathcal{O}$. Note that $\mathcal{A}_V(\mathcal{O})$ can be still be represented on $\mathcal{H}_0$!



2.2 The interacting state

In the next step, one would like to find the ground state $\Omega(g) \in \mathcal{H}_0$ for $H(g)$ in the interacting theory $\mathcal{A}_V(\mathcal{O})$. Haag's theorem does not forbid the existence of such a vector in $\mathcal{H}_0$.
Assume, we have such a $\Omega(g)$, then the Wightman functions of the interacting fields

$$W_n(x_1,\ldots,x_n) = \left\langle \Omega(g) \middle| \Phi(t_1,\vec{x}_1)\cdots\Phi(t_n,\vec{x}_n) \middle| \Omega(g) \right\rangle$$

have the desired properties in a bounded region $\mathcal{O} \subset M$. Notice, that formally all desired properties of a vacuum Wightman QFT are fulfilled formally, if we take $g \to 1$ (adiabatic limit).

However Haag's theorem states here

$$\lim_{g\to 1} \Omega(g) \notin \mathcal{H}_0$$

Nevertheless it is possible, that the W_n converge in the adiabatic limit, defining a new set of Wightman functions. To construct the Hilbert space of the interacting theory, one would have to ensure the existence of the n-point Wightman functions W_n in the sense of distributions. (in $x_1, \ldots, x_n$). Then one can use the Wightman reconstruction theorem, from which one recovers the (interacting) Hilbert space of the theory.

Theorem 2 (Glimm, Jaffe)**.** *The interacting vacuum Wightman functions for Φ^{2n}-interactions exist in 2-dimensional Minkowski space, the Φ^4-theory exists in 3-dim. Minkowski spacetime.*

3 Perturbative approaches

3.1 Interaction picture

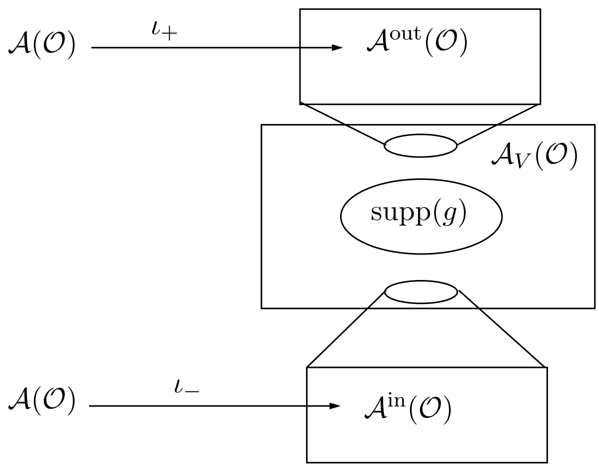
In the traditional approach to perturbative QFT, one constructs the interacting theory out of the free one in a formal power series, with the help of the S-matrix. It heavily relies on the interaction picture of QM, in which the S-matrix is defined as

$$S = \exp_{\cdot_{\mathcal{T}}} \left(-i \int \mathrm{d}^4x \; V(\phi_0(t, \vec{x})) \right)$$

which is not a well-defined operator on $\mathcal{H}_0$ for non-trivial V. However, by inserting the test-function $g \in C_0^\infty(\mathbb{R}^4)$, we view the interaction in terms of time-zero fields as

$$V(g) = \int \mathrm{d}^4x \; g(x) \, \mathbf{:}P(\phi_0(0, \vec{x}))\mathbf{:}$$

as a compactly supported perturbation. Then we can embed two copies of the free theory in the interacting one, if the resp. regions are in the past & future of supp(g), i.e. we have $\iota_\pm : \mathcal{A}(\mathcal{O}) \to \mathcal{A}_V(\mathcal{O})$ for $\mathcal{O}$ in the past/future of supp(g). The picture is the following



In general the $\iota_\pm$ are homomorphism, i.e. not all scattering states can be interpreted as free fields (e.g. formation of bound states). The invertibility of $\iota_\pm$ is called asymptotic completeness. The fields $\iota_\pm(\phi_0) = \Phi^{\text{out/in}}$ represent the free fields in the interacting theory. In particular we have for every g

$$
\begin{aligned}
\Phi(0,\vec{x}) &= \lim_{s\to-\infty} \mathrm{e}^{-iH(g)s}\mathrm{e}^{iH_0s}\Phi^{\mathrm{in}}(0,\vec{x})\mathrm{e}^{-iH_0s}\mathrm{e}^{iH(g)s} \\
\Phi(t,\vec{x}) &= \mathrm{e}^{iH(g)t}\lim_{s\to-\infty} \mathrm{e}^{-iH(g)s}\mathrm{e}^{iH_0s}\Phi^{\mathrm{in}}(0,\vec{x})\mathrm{e}^{-iH_0s}\mathrm{e}^{iH(g)s}\mathrm{e}^{-iH(g)t} \\
&= \lim_{t\to-\infty} \mathrm{e}^{-iH(g)t}\mathrm{e}^{iH_0t}\Phi^{\mathrm{in}}(t,\vec{x})\mathrm{e}^{-iH_0t}\mathrm{e}^{iH(g)t}
\end{aligned}
$$

which can be written in a perturbative expansion

$$
\Phi(t,\vec{x}) = U_g(t,-\infty)^{-1}\Phi^{\mathrm{in}}(t,\vec{x})U_g(t,-\infty)
$$

with

$$
U_g(t,s) = \exp_{\cdot_{\mathcal{T}}}\left(-i\int_s^t \mathrm{d}t'\ \mathrm{e}^{it'H_0}V(g)\mathrm{e}^{-it'H_0}\right) \qquad t > s
$$

This is rewritten with the formal S-matrix $S(g) = U_g(\infty,-\infty)$ by

$$
\Phi(t,\vec{x}) = U_g(\infty,-\infty)^{-1}U_g(\infty,t)\Phi^{\mathrm{in}}(t,\vec{x})U_g(t,-\infty) = S(g)^{-1}(S(g)\cdot_{\mathcal{T}}\Phi^{\mathrm{in}}(t,\vec{x})) = R_{V_g}(\Phi^{\mathrm{in}}(t,\vec{x}))
$$

R_{V_g} can be given as a formal power series (in g) in terms of free fields Φ^{in}. The vacuum state in $\mathcal{A}^{\mathrm{in}}(\mathcal{O}) = \iota_-(\mathcal{A}(\mathcal{O}))$ extends to some (non-translation-invariant) state on $\mathcal{A}_V(\mathcal{O})$. It can be shown, that in the adiabatic limit, it tends to the *unique vacuum on* $\mathcal{H}$. Taking this into account, we are left with the following expression for the Wightman functions

$$
\begin{aligned}
W_n(x_1,\ldots,x_n) &= \lim_{g\to1}\langle\Omega_0|U_g(-\infty,t_1)\phi_0(t_1,\vec{x}_1)U(t_1,t_2)\cdots\phi_0(t_n,\vec{x}_n)U_g(t_n,-\infty)|\Omega_0\rangle \\
&= \lim_{g\to1}\langle\Omega_0|\underbrace{U_g(-\infty,\infty)}_{S_g^{-1}}U_g(\infty,t_1)\phi_0(t_1,\vec{x}_1)U(t_1,t_2)\cdots\phi_0(t_n,\vec{x}_n)U_g(t_n,-\infty)|\Omega_0\rangle
\end{aligned}
$$

which agrees with the Gell-Mann and Low formula, if all time arguments of W_n are ordered and the usual gymnastics are taken to bring S_g^{-1} in the denominator.