AQFT

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1 QFT on Minkowski spacetime

1.1 Drawbacks of the Hilbert space description of QFT

One theory, one Hilbert space? In QM, every representation of the CCR $[x, p] = i\hbar$ on a Hilbert space $\mathcal{H}$ is unitary equivalent to the Schrödinger representation (Stone-von-Neumann theorem)

And in QFT? Not only does the theorem not hold for the CCR

$$[\phi(x^0, \boldsymbol{x}), \dot{\phi}(x^0, \boldsymbol{y})] = i\hbar\delta(\boldsymbol{x} - \boldsymbol{y})$$

the bug of inequivalent representations is a feature, since two states, which differ by an infinite amount of energy yield inequivalent representations

Perturbation theory Having fixed a Hilbert space $\mathcal{H}$ and representations of the CCR and the translation group (with generators $P_\mu = (H, \boldsymbol{P})$. Can one apply the QM formulation of PT? **HAAGS THEOREM**: The interaction picture does not exist, i.e. all other Hamiltonians H', which are translation invariant and define an interacting time-evolution by

$$\mathrm{e}^{iH't}\phi(0, \boldsymbol{x})\mathrm{e}^{iH't}$$

for the time-zero fields $\phi(0, \boldsymbol{x})$ are proportional to the free Hamiltonian $H' = H +$ const.

And path-integrals? The Euclidean measure of the interacting theory, defined by

$$\mathrm{d}\mu(\phi) = Z^{-1}\mathrm{e}^{i\int \mathrm{d}^4x\, V(\phi(x))}\,\mathrm{d}\mu_0(\phi)$$

exists only, if V is at most quadratic.

1.2 Systematic approach by AQFT

1.2.1 Straightforward approach to CM

What means mechanics? Want to describe the dynamics of pointlike (test-)particles under the influence of internal and external forces

Specify the arena for physics! Specify a configuration space Q, i.e. the space of all possible positions of the particles. Usually $Q = S^N$ where S is a subset of $\mathbb{R}^3$ and $N = \#$ of particles

Specify the model! Introduce a function H on T^*Q, which describes the total energy of the system, depending on all positions and momenta (and possibliy time)

Which dynamics is realized? Search for solutions of Hamilton's equations: $\dot{q} = \nabla_p H$ and $\dot{p} = -\nabla_q H$ for given ICs (q_0, p_0) **STATES**

What can be measured? Real functions $F \in C^\infty(T^*Q)$ can be used to model a measurement apparatus. The evluation of F on the specific solution yields the predicted measured value, given the ICs **OBSERVABLES**

1.2.2 Geometric approach to CM

Many physical situations yield similar properties, so can we find a description of CM, which gives us information about general systems, without referring to specific solutions?

What do all observables have in common? The sum/product of observables are observables again. $\mathcal{A}(Q) = C^\infty(T^*Q)$ is a commutative algebra.

Relations between observables? There is a unique Poisson bracket $\{\cdot, \cdot\}$ for every Hamiltonian H, which

- is antisymmetric
- acts as derviation in both entries
- fulfils the Jacobi-identity $\rightarrow$ Lie-algebra
- imposes dynamical equations for the observables $\dot{F} = \{H, F\}$

independent of the particular solution. **POISSON ALGEBRA**

States The pure states of the system are evaluation functionals on solutions

2 Relativistic Field theory on $\mathbb{R}^4$

Observables in field theory: Fields at a point x: $\phi(x)$ is interpreted as field strength at a point x. In quantum theory: impossible to measure ϕ at a sharp point x due to Heisenberg principle $\rightarrow \phi(x)$ is much too singular

Smeared fields: Go over to a mean of ϕ over a region, weighted with a weight f. Then

$$\phi(f) = \int \mathrm{d}x \; f(x)\phi(x)$$

represents this quantity. Conveniently we choose $\operatorname{supp} f$ bounded, then ϕ becomes a distribution on M.

What is f? The smearing function f models the measurement apparatus, which measures ϕ in a compact region in space and time

New paradigm: The fundamental object of the theory is the set $\{\mathcal{A}(\mathcal{O}) : \mathcal{O} \subset M\}$ of local algebras. $\mathcal{A}(\mathcal{O})$ is generated by smeared fields with support in $\mathcal{O}$:

$$\mathcal{A}(\mathcal{O}) = \bigcup_{N=1}^{\infty} \left\{ \sum_{n=1}^{N} \lambda_n \phi(f_1) \cdots \phi(f_n), \quad \operatorname{supp}(f_i) \subset \mathcal{O} \right\}$$

Expected relations: If $\mathcal{A}(\mathcal{O})$ shall encode all possible observables in $\mathcal{O}$, then we expect

- Causality: If $\mathcal{O}$ is spacelike to $\mathcal{O}'$, then $[\mathcal{A}(\mathcal{O}), \mathcal{A}(\mathcal{O}')] = 0$
- Compatibility (Isotony): If $\mathcal{O} \subset \mathcal{O}'$ then $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')$
- Covariance: $\mathfrak{p}$ Poincare trafo :$\mathcal{O} \rightarrow \mathcal{O}'$, then we have an homorphism of $\alpha_{\mathfrak{p}} : \mathcal{A}(\mathcal{O}) \rightarrow \mathcal{A}(\mathfrak{p}\mathcal{O})$
- Time-slice axiom: The algebra $\mathcal{A}(\mathcal{O})$ is contained in $\mathcal{A}(N_\varepsilon(\Sigma))$ where Σ is a Cauchy surface for $\mathcal{O}$ and $N_\varepsilon(\Sigma)$ is an ε-nbhd. of Σ: Existence of hyperbolic equations
- (optional) Existence of a ground state: Not fundamental object and not important for theory

which are in fact the **HAAG-KASTLER AXIOMS**

Aha... Formalism is independent of a concrete realisation of $\mathcal{A}(\mathcal{O})$ as operators on a Hilbert space.

States on $\mathcal{A}(\mathcal{O})$ A state ω on $\mathcal{A}(\mathcal{O})$ is a positive (continuous) linear functional on $\mathcal{A}(\mathcal{O})$, such that $\omega(A^*A) \geq 0$ and $\omega(\mathbb{1}) = 1$. On a Hilbert space every vector and every density matrix yield a state by

$$\omega(A) = \langle\psi|\, A\, |\psi\rangle\,, \qquad \omega(A) = \mathrm{tr}(\varrho A)$$

In general the space of states contains a lot of unphysical states (selection criterion needed)

Where is my Hilbert space? Fixing a state ω on $\mathcal{A}(\mathcal{O})$, then there is a representation of $\mathcal{A}(\mathcal{O})$ as operators on a Hilbert space $\mathcal{H}$, which is unique up to unitary equivalence **GNS CONSTRUCTION**

$$(\mathcal{A}(\mathcal{O}), \omega) \stackrel{\mathrm{GNS}}{\longrightarrow} (\pi, \mathcal{H}, \Omega), \quad \pi : \mathcal{A}(\mathcal{O}) \to \mathrm{LinOp}(\mathcal{H}), \quad \Omega \in \mathcal{H}$$

with $\langle\Omega|\, \pi(A)\, |\Omega\rangle = \omega(A)$

Where are my beloved particles? Depends on ω, if ω is pure, then there is a Hilbert space $\mathcal{H}_1$, such that the GNS-Hilbert space is the Fock space over $\mathcal{H}_1$ with Fock-vacuum Ω.

Examples

- Free scalar field, vacuum state specified by

$$\omega_{\mathrm{vac}}(\phi(x)\phi(y)) = \frac{1}{(2\pi)^3} \int \mathrm{d}^4 p\ \mathrm{e}^{-ip(x-y)} \theta(p_0)\delta(p^2 - m^2)$$

  and higher moments by quasifree-property is pure. Reduces to usual constructions, including creation and annihilation operators.

- Free scalar field, thermal equilibrium state, defined by

$$\omega_\beta(\phi(x)\phi(y)) = \frac{1}{(2\pi)^3} \int \mathrm{d}^4 p\ \mathrm{e}^{-ip(x-y)} \frac{\varepsilon(p_0)\delta(p^2 - m^2)}{1 - \mathrm{e}^{-\beta p_0}}$$

  ω_β is mixed, GNS Hilbert space $\mathcal{H}$ is no Fock space, no stable excitation from Ω exists.

3 Interacting QFT

3.1 The algebra of Wick products

Local interactions Can we add local interactions to $\mathcal{A}(\mathcal{O})$? Experience shows, that they provide physical meaningful models. Look at

$$\phi(f)\phi(g) = \int \mathrm{d}x \ \mathrm{d}y \ \phi(x)\phi(y)f(x)g(y)$$

We would need $f(x)g(y) \to f(x)\delta(x-y)$ for powers of ϕ at the same point. Then

$$\lim_{f\cdot g \to f \cdot \delta} \phi(f)\phi(g) = \int \mathrm{d}x \ \phi^2(x)f(x)$$

This is not possible, the pointwise product $\phi(x)\phi(y)$ is way to singular for such an operation.

Wick ordering Define a H-dependent Wick-ordering by

$$:\!\phi(f)\phi(g)\!:_H = \phi(f)\phi(g) - H(f,g)\mathbb{1}$$

and higher products accordingly. One can take the above limit, hence define objects like $:\!\phi^n(f)\!:_H$, whenever H "kills the singularity on the diagonal"

Hadamard distribution Every such H is of the form

$$H(x,y) \sim \frac{1}{(x-y)^2 + i\varepsilon(x^0-y^0)} - \frac{m^2}{2}\log\left[m^2(x-y)^2\right] + \mathrm{smooth}(x,y)$$

where H has to be a bisolution to hyperbolic equation of motion for $\phi(f)$. There exist many of such – equivalent – Wick orderings. One distinguished H on Minkowski by Poincare invariance $\to$ vacuum two-point function $H = \Delta_+$

Algebra of Wick polynomials $\mathcal{W}_H$ contains polynomials of the field by

$$:\!\Phi^n(f)\!:_H = \int \mathrm{d}x \ :\!\phi(x)^n\!:_H f(x)$$

and depends on H. The extension of the operator product on $\mathcal{A}(\mathcal{O})$ yields Wicks theorem

$$:\Phi^n(f):_H :\Phi^n(g):_H = \sum_{k=1}^{n} \int \mathrm{d}x\ \mathrm{d}y\ :\phi^{n-k}(x)\phi^{n-k}(y):_H H(x,y)^k f(x)g(y)$$

where the k-fold power of H at the same point is well-defined, due to the singular structure of H, which behaves well enough

Time-ordered products A time-ordered product for elements of $\mathcal{A}(\mathcal{O})$ can easily obtained by

$$\phi(f) \cdot_{\mathcal{T}} \phi(g) = \int \mathrm{d}x\ \mathrm{d}y\ f(x)g(y)\big(\phi(x)\phi(y) - \underbrace{\theta(y^0 - x^0)[\phi(x), \phi(y)]}_{i\Delta_{\mathrm{adv}}}\big)$$

Time-ordering on $\mathcal{W}_H$ Trying to accomplish the same structure on $\mathcal{W}_H$ amounts to replacing $H \to H - i\Delta_{\mathrm{adv}} = H_F$ in Wicks theorem

$$:\Phi^n(f):_H \cdot_{\mathcal{T}} :\Phi^n(g):_H = \sum_{k=1}^{n} \int \mathrm{d}x\ \mathrm{d}y\ :\phi^{n-k}(x)\phi^{n-k}(y):_H H_F(x,y)^k f(x)g(y)$$

The k-fold power of H_F is unfortunately not a well defined distribution: well-known **UV-divergencies of pert. QFT**, since

$$\mathcal{F}(H_F(x-y)^k)(k) = \int \mathrm{d}^{k-1}P\ \tilde{H}_F(p_1)\cdots H_F(p_{k-1})H_F(k - p_1 - \ldots - p_{k-1})$$

Extension of Distributions Due to the time-ordering property of $\cdot_{\mathcal{T}}$, the time-ordered product is defined everywhere, up to the diagonal $x = y$

$$:\Phi^n(f):_H \cdot_{\mathcal{T}} :\Phi^n(g):_H = \begin{cases} :\Phi^n(x):_H :\Phi^n(y):_H & x^0 > y^0 \\ :\Phi^n(y):_H :\Phi^n(x):_H & y^0 > x^0 \end{cases}$$

The extension to $x = y$ is not unique: **Renormalization freedom**

Renormalized time-ordered product: Fixing all free parameters of the extension process (by experimental input) the obtained time-ordered product is called renormalized time-ordered product $\cdot_{\mathcal{T},\mathrm{ren}}$ which is now defined on $\mathcal{W}_H$

3.2 Interacting theory

S-matrix The S-matrix is given by the renormalized time-ordered exponential of a Wick-monomial $V = {:}\Phi^n{:}_H \in \mathcal{W}_H$

$$S(V) = \exp_{\cdot_{\mathcal{T},\text{ren}}}(iV)$$

The main theorem of renormalization states, that every two S-matrices S_1 and S_2, obtained by different renormalizations are related by

$$S_1(V) = S_2(V + A) \qquad A \in \mathcal{W}_H$$

Equivalence of all (known) renormalization prescriptions. A theory is called renormalizable if A has only finitly many monomials, then one needs finitely many parameters at every order to specify the theory.

Interacting algbra The interacting field

$$R_V(\phi(f)) = S(V)^{-1}(S(V) \cdot_{\mathcal{T},\text{ren}} \phi(f))$$

satisfies the interacting field equations and generates the algebra of interacting fields $\mathcal{A}_V(\mathcal{O}) = R_V(\mathcal{A}(\mathcal{O}))$.

Interacting Greens functions and Wightman functions One obtains the interacting Greens- and Wightman functions by

$$\omega(R_V(\phi(f_1) \cdot_{\mathcal{T}} \cdots \cdot_{\mathcal{T}} \phi(f_n)) = \omega(S(V)^{-1}(S(V) \cdot_{\mathcal{T}} \phi(f_1) \cdots \cdot_{\mathcal{T}} \phi(f_n)))$$
$$\omega(R_V(\phi(f_1)) \cdots R_V(\phi(f_n))) = \omega(S(V)^{-1}(S(V) \cdot_{\mathcal{T}} \phi(f_1))S(V)^{-1}(S(V) \cdot_{\mathcal{T}} \phi(f_2)) \cdots)$$

3.3 Adiabatic limit

Adiabatic Limit For the procedure to be well-defined, we had to smear the interaction $V = {:}\Phi^n(f){:}$ with a test function f. Then the above functions are not translation invariant: We need to take the limit $f \to$ const. This is highly risky and does not exist in general (**IR DIVERGENCIES**)

- vacuum in purely massive theories: Epstein and Glaser
- vacuum in massless scalar theories and QED: Blanchard and Seneor
- vacuum in Non-abelian Yang-Mills: unknown (probl. related to mass gap)
- thermal equilibrium state: work in progress

In the adiabatic limit, they states defined above converge to the physical important states (interacting vacuum, interacting thermal equilibrium).

Gell-Mann and Low The Gell-Mann and Low formula for interacting Greens functions can be proven to hold, if we choose H as the vacuum 2PF and ω as the free vacuum state. Then $H_F = \Delta_F$ and we get

$$\omega(S(V)^{-1}(S(V) \cdot_{\mathcal{T}} \phi(f_1) \cdots \cdot_{\mathcal{T}} \phi(f_n))) = \frac{\omega(S(V) \cdot_{\mathcal{T}} \phi(f_1) \cdots \cdot_{\mathcal{T}} \phi(f_n))}{\omega(S(V))}$$

that means, that the derived Feynman rules for vacuum QFT are correct.